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ABSTRACT

Imagery use in high school mathematics classrooms was studied. A visual image was defined as a mental scheme depicting visual or spatial information, but this definition was not spelled out to teachers or students, in order to learn what they meant by the concept. Subjects were 13 high school teachers and 54 of their students interviewed over 3 years. Images used by the students were classified in the following categories: (1) concrete, pictorial imagery; (2) pattern imagery; (3) memory images of formulae; (4) kinesthetic imagery; and (5) dynamic imagery. Images were used as prototypes, as metaphors, and as metonymies (i.e., a subcategory or submodel used to comprehend the category as a whole). Concrete imagery was particularly used in these ways. Pattern imagery, in which concrete details are disregarded and pure relationships are depicted in a visual-spatial scheme, was used much less often. Teachers were classified into three groups, depending on their use of imagery. Very visual teachers often were unaware of difficulties some students had with a visual approach, and non-visual teachers similarly ignored the benefits some students could derive. However, teachers in the middle group, who used some visual aspects, but valued abstraction and generalization, were able to use imagery most effectively. One of four figures illustrates a visual problem solution. (Contains 11 references.) (SLD)

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ED 363 300

Uses and Values of Prototypic Visual Images in High School Mathematics

by
Norma C. Presmeg

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Introduction by way of a problem

In mathematics at all levels and in all branches, some people would consider imagery of various types to be an essential component. This depends, of course, on what they consider imagery to be and on their definitions of mathematics. In the writer's research on imagery use in high school mathematics classrooms a definition was used which was deliberately broad enough to encompass imagery ranging from clear 'pictures in the mind' to vague spatial intuitions. A visual image was defined as "a mental scheme depicting visual or spatial information" (Presmeg, 1985, p. 90). This definition was never spelled out to the teachers and students in the research, because discussion of their imagery in context (in the course of several hundred transcribed interviews) enabled the researcher to interpret to some extent what they meant by the construct, 'image'. Of all the possible modalities, it was predominantly the visual one which assumed significance in these studies, although kinesthetic imagery was also reported occasionally and thus formed a category along with four other types of imagery (all visual).

By way of introduction to the uses and values of visual imagery in mathematics, consider the following problem (Krutetskii, 1976, p. 159).

A train passes a telegraph pole in 1/4 minute, and in 3/4 minute it passes completely through a tunnel 540 meters long. What is the train's speed in meters per minute and its length in meters?

(The reader might like to attempt the problem before reading on.)

This problem was included in the teachers' section of the writer's test for preference for visual modes of processing in mathematics, which was used to select mathematics teachers of various preferences, as well as visualizers in their classes, to participate in the research. Many teachers used algebra to solve the problem, often in a lengthy process. One teacher gave the elegant visual solution shown in figure 1, which speaks for itself.

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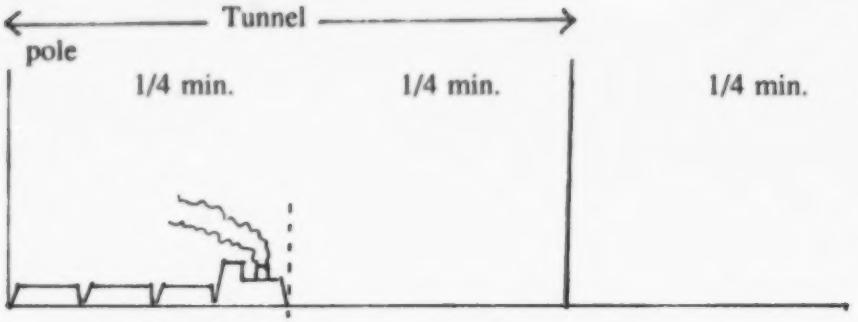

After 1/4 minute train is in tunnel as shown in sketch.
Therefore length of train is 270 meters.
Speed of train is (270 x 4) = 1080 meters per minute.

Figure 1. Rough sketch. Teacher's solution to tunnel problem.

Notice that logical reasoning is also an integral part of this solution. Following Krutetskii (1976), the position was taken that all mathematics involves reasoning and logic. The visuality of a solution to a problem, then, was judged by the presence of a diagram or report of an image in the accompanying self-report questionnaire. As interviews demonstrated, there was tremendous variation in visual modes of processing throughout the studies. Imagery is certainly personal and idiosyncratic, and some forms may be considered more 'abstract' than others, as will be shown. In the case of the above problem and its visual solution, the writer displayed the statement of the problem and the diagram, with the artist's permission, at a talk given to a group of teachers. One member of the audience, who was in the research project, spoke rather scathingly in an interview later, as follows.

MRS GREEN (pseudonym): You know, it rather depends on what you mean by a visual image. An actual picture; well to my mind that thing with the train going into the tunnel and sticking half out of the tunnel and that ... to my mind [laughing] that was utterly ridiculous! It was quite un-necessary. I mean, I'd never dream of actually drawing the train.

I (interviewer): One could do a line.

MRS GREEN: I would put some sort of symbols, possibly, but I'd never get as far as that. My idea of an image is more or less a diagrammatic thing, and I like to show children, er, ... geometry lends itself to this kind of thing and of course algebraic problems and so on. I like to show them a sort of progression of steps. I very often give them a sort of summarized, mode of approach in problems; you know, translate from the words into algebraic symbols, analyzing each step of the information; that sort of thing. Now that, to my mind, is a visual image.

I: That is just as visual. Categorize and make a table [an aspect of MRS GREEN's teaching which had been observed on several

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occasions].

MRS GREEN: That's right. I tabulate, rather, and make diagrams rather than ... I mean actual little drawings of trains going into tunnels [laughs]; I haven't got time for that.

(Presmeg, 1985, p.220; "..." indicates pauses, not that segments of the interview are omitted.)

In this extract, MRS GREEN appears to be aware of a more abstract form of visuality which she attempted to convey in her teaching.

Images as mathematical prototypes

In the writer's initial three-year study of imagery in high school mathematics, the images used by the visualizers in the study were classified as follows. The numerals in parentheses indicate the numbers of students, of the 54 in the study, who used these kinds of images when solving high school mathematics problems in the interviews.

concrete, pictorial imagery	(52)
pattern imagery	(18)
memory images of formulae	(32)
kinaesthetic imagery	(16)
dynamic imagery	(2)

(Presmeg, 1985, p. 174).

Examples of all these categories have been published elsewhere (Presmeg, 1986a), and some instances of prototypic usages are illustrated below. Suffice it to say here that the difficulties which these visualizers experienced in their use of imagery and diagrams in mathematics were largely related to the one-case concreteness of the first category. A 'picture', whether external or internal, by its nature is a particular case, and this fact may prevent abstraction and generalization if the essential features cannot be separated from the random ones. But as Bartlett wrote in 1932, the disadvantages of use of imagery are but "the price of its peculiar excellences" (Bartlett, 1977, p. 220). By using imagery in prototypic ways which included metaphors and metonymies, the same features which presented potential difficulties were transmuted into strengths by some of the visualizers in the studies.

The essence of a *prototype* is that it is a mental representation which is a good example of a category (Lakoff, 1987, p. 43). In his extensive discussion of categorization theories, Lakoff described a radial structure in which the prototype lies at the center of a category. On the one hand, *metaphors* may be used to extend this structure or to link it with other categories through the identification of elements in disparate domains. On the other hand, *metonymies* are also valuable elements of categories, whether or not they are prototypic, i.e., at the center of the structure. Lakoff (1987, p. 79) defined a metonymy as,

> *"a situation in which some subcategory or member or submodel is used (often for some limited and immediate purpose) tocomprehend the category as a whole."*

Now, every time a diagram or a visual image is used effectively in mathematics, such a situation is brought into play, i.e., a metonymy is used.

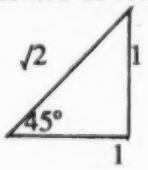
Illustrations

In all five categories of imagery metonymies could be identified, but their use was particularly striking in the cases of concrete images and memory images of formulae. All the following illustrations are from Presmeg (1985).

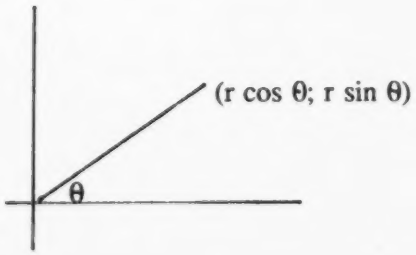
CARON W was a student who had particularly vivid visual images.

CARON W: Cosine 45 equals ... one over root two.
I: Now that looked as if you had a picture in your mind.
CARON W: Yeah, it was the triangle.
I: So you got a picture of the 45 degree triangle, and you can just read it from your image?
CARON W: Yeah.

In common with many other visualizers, CARON could see the prototypic image represented as follows:



This instance is prototypic because it lies at the center in the radial structure of the category of all 45 degree right triangles. Another metonymic prototype in trigonometry was used by JENI D:



Standard diagrams of theorems in geometry often served the same function:

In the category of memory images of formulae, some formulae appeared to have a particular visual impact which facilitated the mnemonic use of an image, as in the following examples:

$$|\underline{AB}| = \sqrt{x^2 + y^2}\,; \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\,;$$

x^2 - (sum of roots) x + (product of roots) = 0.

The more vivid the image, the more facilitative it was of accurate recall. These concrete images ("I saw a page in my book" - CARON W) are metonymic in the sense that they frequently stand for an abstract procedure or principle which is recalled by means of the image.

Concrete imagery may also be used metaphorically, as in the following extract from an interview in which ALISON M was working with trigonometric ratios of angles in the four quadrants:

ALISON M: Then it would be sine ... second quadrant. So ... 180; you take it from 180, because that's your water level.

I: Oh, is that how you think of it? And how does the water level help you to get it?

ALISON M: Oh, um ... you've got ... It's like a ship sailing: can't sail that way really [up and down, i.e., y axis]. There, that, sort of ... it can. You work it from there. You take most of your ... 'cause that's your 360 ... it's there as well.

I: Did you think of that for yourself or did someone tell you about the water level?

ALISON M: No, I just thought of it.

ALISON's vivid and idiosyncratic imagery of a ship sailing and a water level was metaphorically used to remind her to use 180 degrees and 360 degrees, and not 90 degrees and 270 degrees, when calculating the acute angle she needed to find the trigonometric ratios. This is an instance of metaphoric use of concrete imagery. Kinesthetic imagery was also used in this way, for example in the case of several students who "walked" with their fingers to show four or five vectors head-to-tail when illustrating the concept of displacement for the researcher. A mixture of kinesthetic and dynamic imagery was used metaphorically by ISABEL M, who had sketched a linear graph after working out the coordinates of two points:

I: Would you normally have just two points, or would you work out a third one?

ISABEL M: Well you can work out a third one. MISS MAUVE [pseudonym] said that you can plot by two. If your points are close, then take a third one, because there's not so much movement, you know [illustrating with her hand the movement of the line in a see-saw motion]. If they're close, there's a lot of movement.

The implicit see-saw metaphor gave ISABEL the rationale for the number of points whose coordinates she calculated.

Some uses and values of various kinds of imagery used in metaphoric and metonymic ways have been illustrated. There is another type of imagery which is particularly important in mathematical reasoning and concept formation. These images were called by the writer, "pattern images", and by Johnson (1987), "image-schematic structures".

Pattern imagery

Reasoning, abstraction and generalization are necessary components of mathematical thought (Davis and Hersh, 1981). One way that imagery may be used in these processes is by making a concrete image the bearer of abstract information (Neisser, 1967), as illustrated in the foregoing metaphoric and metonymic usages. There is a second way, which is more difficult to illustrate because the images used are not like 'pictures'. An image itself may be sketchy and abstract; a method of working has a certain 'shape' to it which may be visualized (Arnheim, 1969). The type of imagery in which concrete details are disregarded and pure relationships are depicted in a visual-spatial scheme was designated *pattern imagery* in the writer's studies. These pattern images bear a striking resemblance to Johnson's *image-schematic structures*, which he described as mediating between images or objects of sensation, on the one hand, and abstract concepts, on the other (1987, p. 155). Johnson did not consider these

structures to be images; but then, his definition of imagery appears to have been rather narrow, confined to what the writer would describe as *concrete imagery*.

The value of Johnson's account for mathematics education lies in the explicit recognition that "imagination", supported by image-schematic structures and used in a variety of ways including metaphors and metonymies, may be an integral part of the reasoning process itself. The writer's research provides some support for this position (Presmeg, 1992). Of the 54 visualizers in the initial study, only two, CRISPIN T and MARA L, consistently used more abstract forms of imagery, including pattern imagery. Both these students were in the classes of teachers who stressed the value of abstraction, generalization and recognition of patterns. MARA was in the mathematics class of MRS GREEN, whose comments on more abstract forms of imagery were quoted at the beginning of this paper in the context of the train problem. CRISPIN was in the class of MR BLUE, who stressed patterns and regularities in his teaching, although his mathematical problem solving style (as distinct from his teaching style) was singularly nonvisual. It is noteworthy that CRISPIN and MARA were the only two students in this study who achieved highly in their school-leaving mathematics examinations (Presmeg, 1986b). The attitudes which their mathematics teachers held towards the uses and values of imagery of various kinds apparently played a part in this achievement.

Teacher beliefs about imagery

The thirteen teachers in the initial study were arranged in three groups, i.e., visual, middle, and nonvisual, on the basis of their "teaching visuality", a construct which provided a ranking of the teachers in terms of their use of literature-based classroom aspects facilitating visual imagery. It may be noted in passing that *teaching visuality* was only weakly correlated with these teachers' *mathematical visuality*, i.e., the visuality of their mathematical problem solving styles (Spearman's $r = 0.404$, n.s.). For the present discussion, it is of relevance that the three groups of teachers held very different beliefs about imagery and its values in learning mathematics. Teachers in the visual group were unanimously positive about the visual methods which they employed in their classrooms, but they were seldom aware of the potential pitfalls (e.g., the one-case concreteness) of some kinds of imagery or graphic representations. In the nonvisual group, teachers typically avoided visual aspects as far as possible and consequently assigned little value to these. Visualizers in the classes of these teachers often came to distrust their own preferred visual methods of processing; only if they were able to relinquish their visual methods in favor of methods consonant with their teachers' could these students do (relatively) well. It was the teachers in the middle group, who used some visual aspects in their teaching but valued abstraction and generalization, whose students were able to use consistently, forms of imagery which overcame some of the potential limitations.

It is the writer's belief that increased awareness of different forms of imagery and their uses and values in the mathematics classroom would result in teaching which facilitated more effective use of this potentially powerful mode of learning mathematics.

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